Properties of Operators

1) The inverse of $\hat{A}$ (written $\hat{A}^{-1}$) is defined by

$$\hat{A}^{-1} \hat{A} = \hat{A} \hat{A}^{-1} = \mathbf{I} \quad (1)$$

2) The transpose of $\hat{A}$ (written $\hat{A}^T$) is a matrix with elements inverted about the diagonal

$$\left(\hat{A}^T\right)_{qn} = A_{qn} \quad (2)$$

If $\hat{A}^T = -\hat{A}$ then the matrix is antisymmetric.

3) The trace of $\hat{A}$ is defined as the sum over the diagonal elements of $\hat{A}$

$$Tr(\hat{A}) = \sum_q A_{qq} \quad (3)$$

4) The Hermitian Adjoint of $\hat{A}$, written $\hat{A}^\dagger$, is

$$\hat{A}^\dagger = \left(\hat{A}^*\right)^\dagger \quad (4)$$

$$\left(\hat{A}^*\right)_{qn} = \left(\hat{A}_{qn}\right)^*$$

5) $\hat{A}$ is Hermitian if

$$\hat{A}^\dagger = \hat{A} \quad \left(\hat{A}^\dagger\right)^* = A \quad (5)$$

If $\hat{A}$ is Hermitian, then $\hat{A}^\dagger$ is Hermitian and $e^{\hat{A}}$ is Hermitian. For a Hermitian operator, $\langle\psi | \hat{A} \phi\rangle = \langle\psi | \hat{A}^\dagger \phi\rangle$. Expectation values of Hermitian operators are real, so all physical observables are associated with Hermitian operators.

6) $\hat{A}$ is a unitary operator if the Hermitian adjoint is also the inverse operator

$$\hat{A}^\dagger = \hat{A}^{-1} \quad \left(\hat{A}^\dagger\right)^* = \hat{A}^{-1} \quad (6)$$

$$\hat{A} \hat{A}^\dagger = \mathbf{I} \quad \Rightarrow \quad \left(\hat{A}^\dagger\right)_{qn} = \delta_{qn}$$

If $\hat{A}$ is Hermitian, then $e^{i\hat{A}}$ is unitary.