EINSTEIN B COEFFICIENT AND ABSORPTION CROSS-SECTION

The rate of absorption induced by the field is

\[
\omega_{k\ell} (\omega) = \frac{\pi}{2\hbar^2} |E_0(\omega)|^2 \left| \langle k | \hat{\epsilon} \cdot \vec{\mu} | \ell \rangle \right|^2 \delta (\omega_{k\ell} - \omega)
\]  

(1)

The rate is clearly dependent on the strength of the field. The variable that you can most easily measure is the intensity \(I\), the energy flux through a unit area, which is the time-averaged value of the Poynting vector, \(S\)

\[
S = \frac{c}{4\pi} (\vec{E} \times \vec{B})
\]  

(2)

\[
I = \langle S \rangle = \frac{c}{4\pi} \langle \vec{E}^2 \rangle = \frac{c}{8\pi} E_0^2.
\]  

(3)

Using this we can write

\[
\omega_{k\ell} = \frac{4\pi^2}{3c\hbar^2} I(\omega) \left| \langle k | \hat{\epsilon} \cdot \vec{\mu} | \ell \rangle \right|^2 \delta (\omega_{k\ell} - \omega),
\]  

(4)

where I have also made use of the uniform distribution of polarizations applicable to an isotropic field: \(\vec{E}_0 \cdot \hat{x} = |\vec{E}_0 \cdot \hat{y}| = |\vec{E}_0 \cdot \hat{z}| = \frac{1}{3} |\vec{E}_0|^2\). An equivalent representation of the amplitude of a monochromatic field is the energy density

\[
U = \frac{I}{c} = \frac{1}{8\pi} E_0^2.
\]  

(5)

which allows the rates of transition to be written as

\[
\omega_{k\ell} = B_{k\ell} U (\omega_{k\ell})
\]  

(6)

The first factor contains the terms in the matter that dictate the absorption rate. \(B\) is independent of the properties of the field and is called the Einstein B coefficient

\[
B_{k\ell} = \frac{4\pi^2}{3\hbar^2} |\mu_{k\ell}|^2.
\]  

(7)

You may see this written elsewhere as \(B_{k\ell} = \left(2\pi/3\hbar^2\right) |\mu_{k\ell}|^2\), which holds when the energy density of a wave is expressed in Hz instead of angular frequency.
If we associate the energy density with a number of photons $N$, then $U$ can also be written in a quantum form

$$N\hbar\omega = \frac{E_0^2}{8\pi}, \quad U = N\frac{\hbar\omega^3}{\pi^2c^3}. \quad (8)$$

Now let’s relate the rates of absorption to a quantity that is directly measured, an absorption cross-section $\alpha$:

$$\alpha = \frac{\text{total energy absorbed / unit time}}{\text{total incident intensity (energy / unit time / area)}}$$

$$= \frac{\hbar\omega \cdot w_{kk}}{I} = \frac{\hbar\omega \cdot B_{kk} U(\omega_{kk})}{cU(\omega_{kk})}$$

$$= \frac{4\pi^2}{\hbar c} |\mu_{kk}|^2 = \frac{\hbar\omega}{c} B_{kk}$$

(9)

More generally, you may have a frequency-dependent absorption coefficient $\alpha(\omega) \propto B_{kk}(\omega) = B_{kk} g(\omega)$ where $g(\omega)$ is a unit normalized lineshape function. The golden rule rate for absorption also gives the same rate for stimulated emission. Given two levels $|m\rangle$ and $|n\rangle$:

$$w_{nm} = w_{nm}$$

$$B_{nm} U(\omega_{nm}) = B_{nm} U(\omega_{nn}) \quad \text{since } U(\omega_{nm}) = U(\omega_{nn})$$

$$B_{nm} = B_{mn}$$

(10)

The absorption probability per unit time equals the stimulated emission probability per unit time. Also, the cross-section for absorption is equal to an equivalent cross-section for stimulated emission, $(\alpha_A)_{nm} = (\alpha_{SE})_{mn}$. 