

REVIEW OF FREE ELECTROMAGNETIC FIELD

Maxwell's Equations in SI units are:

$$(1) \quad \bar{\nabla} \cdot \bar{B} = 0$$

$$(2) \quad \bar{\nabla} \cdot \bar{E} = \rho / \epsilon_0$$

$$(3) \quad \bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$(4) \quad \bar{\nabla} \times \bar{B} = \mu_0 \bar{J} + \epsilon_0 \mu_0 \frac{\partial \bar{E}}{\partial t}$$

\bar{E} : electric field; \bar{B} : magnetic field; \bar{J} : current density; ρ : charge density; ϵ_0 : electrical permittivity; μ_0 : magnetic permittivity.

We are interested in describing \bar{E} and \bar{B} in terms of a scalar and vector potential. This is required for our interaction Hamiltonian.

Generally: A vector field \bar{F} assigns a vector to each point in space. The divergence of the field

$$(5) \quad \bar{\nabla} \cdot \bar{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

is a scalar. For a scalar field ϕ , the gradient

$$(6) \quad \nabla \phi = \frac{\partial \phi}{\partial x} \hat{x} + \frac{\partial \phi}{\partial y} \hat{y} + \frac{\partial \phi}{\partial z} \hat{z}$$

is a vector for the rate of change at one point in space. Here $\hat{x}^2 + \hat{y}^2 + \hat{z}^2 = \hat{r}^2$ are unit vectors.

The Laplacian ∇^2 is the divergence of the gradient of a scalar field $\nabla^2 f = \bar{\nabla} \cdot \nabla f$, or in Cartesian coordinates

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Also, the curl

$$(7) \quad \bar{\nabla} \times \bar{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

is a vector whose x , y , and z components are the circulation of the field about that component.

Some useful identities from vector calculus are:

$$(8) \quad \bar{\nabla} \cdot (\bar{\nabla} \times \bar{F}) = 0$$

$$(9) \quad \nabla \times (\nabla \phi) = 0$$

$$(10) \quad \nabla \times (\bar{\nabla} \times \bar{F}) = \bar{\nabla} (\bar{\nabla} \cdot \bar{F}) - \bar{\nabla}^2 \bar{F}$$

We now introduce a vector potential $\bar{A}(\bar{r}, t)$ and a scalar potential $\phi(\bar{r}, t)$, which we will relate to \bar{E} and \bar{B}

Since $\bar{\nabla} \cdot \bar{B} = 0$ and $\bar{\nabla} (\bar{\nabla} \times \bar{A}) = 0$:

$$(11) \quad \bar{B} = \bar{\nabla} \times \bar{A}$$

Using (3), we have:

$$\bar{\nabla} \times \bar{E} = -\bar{\nabla} \times \frac{\partial \bar{A}}{\partial t}$$

or


$$(12) \quad \bar{\nabla} \times \left[\bar{E} + \frac{\partial \bar{A}}{\partial t} \right] = 0$$

From (9), we see that a scalar product exists with:

$$(13) \quad \bar{E} + \frac{\partial \bar{A}}{\partial t} = -\bar{\nabla} \phi(\bar{r}, t)$$

or

$$(14) \quad \bar{E} = \frac{\partial \bar{A}}{\partial t} - \nabla \phi$$

 convention

So we see that the potentials \bar{A} and ϕ determine the fields \bar{B} and \bar{E} :

$$(15) \quad \bar{B}(\bar{r}, t) = \bar{\nabla} \times \bar{A}(\bar{r}, t)$$

$$(16) \quad \bar{E}(\bar{r}, t) = -\bar{\nabla}\varphi(\bar{r}, t) - \frac{\partial}{\partial t}\bar{A}(\bar{r}, t)$$

We are interested in determining the wave equation for \bar{A} and φ . Using (15) and differentiating (16) and substituting into (4):

$$(17) \quad \bar{\nabla} \times (\bar{\nabla} \times \bar{A}) + \epsilon_0 \mu_0 \left(\frac{\partial^2 \bar{A}}{\partial t^2} + \bar{\nabla} \frac{\partial \varphi}{\partial t} \right) = \mu_0 \bar{J}$$

Using (10):

$$(18) \quad \left[-\bar{\nabla}^2 \bar{A} + \epsilon_0 \mu_0 \frac{\partial^2 \bar{A}}{\partial t^2} \right] + \bar{\nabla} \left(\bar{\nabla} \cdot \bar{A} + \epsilon_0 \mu_0 \frac{\partial \varphi}{\partial t} \right) = \mu_0 \bar{J}$$

From (14), we have:

$$\bar{\nabla} \cdot \bar{E} = -\frac{\partial \bar{\nabla} \cdot \bar{A}}{\partial t} - \bar{\nabla}^2 \varphi$$

and using (2):

$$(19) \quad \frac{-\partial \bar{\nabla} \cdot \bar{A}}{\partial t} - \bar{\nabla}^2 \varphi = \rho / \epsilon_0$$

Notice from (15) and (16) that we only need to specify four field components (A_x, A_y, A_z, φ) to determine all six \bar{E} and \bar{B} components. But \bar{E} and \bar{B} do not uniquely determine \bar{A} and φ . So, we can construct \bar{A} and φ in any number of ways without changing \bar{E} and \bar{B} . Notice that if we change \bar{A} by adding $\bar{\nabla}\chi$ where χ is any function of \bar{r} and t , this won't change \bar{B} ($\nabla \times (\nabla \cdot B) = 0$). It will change E by $(-\frac{\partial}{\partial t} \bar{\nabla}\chi)$, but we can change φ to $\varphi' = \varphi - (\partial\chi/\partial t)$. Then \bar{E} and \bar{B} will both be unchanged. This property of changing representation (gauge) without changing \bar{E} and \bar{B} is gauge invariance. We can transform between gauges with:

$$(20) \quad \bar{A}'(\bar{r}, t) = \bar{A}(\bar{r}, t) + \bar{\nabla} \cdot \chi(\bar{r}, t) \quad \begin{array}{l} \text{gauge} \\ \text{transformation} \end{array}$$

$$(21) \quad \varphi'(\bar{r}, t) = \varphi(\bar{r}, t) - \frac{\partial}{\partial t} \chi(\bar{r}, t)$$

Up to this point, A' and Q are undetermined. Let's choose a χ such that:

$$(22) \quad \bar{\nabla} \cdot \bar{A} + \epsilon_0 \mu_0 \frac{\partial \varphi}{\partial t} = 0 \quad \text{Lorentz condition}$$

then from (17):

$$(23) \quad -\nabla^2 \bar{A} + \epsilon_0 \mu_0 \frac{\partial^2 \bar{A}}{\partial t^2} = \mu_0 \bar{J}$$

The RHS can be set to zero for no currents.

From (19), we have:

$$(24) \quad \epsilon_0 \mu_0 \frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = \frac{\rho}{\epsilon_0}$$

Eqns. (23) and (24) are wave equations for \bar{A} and φ . Within the Lorentz gauge, we can still arbitrarily add another χ (it must only satisfy 22). If we substitute (20) and (21) into (24), we see:

$$(25) \quad \nabla^2 \chi - \epsilon_0 \mu_0 \frac{\partial^2 \chi}{\partial t^2} = 0$$

So we can make further choices/constraints on \bar{A} and φ as long as it obeys (25).

For a field far from charges and currents, $J = 0$ and $\rho = 0$.

$$(26) \quad -\bar{\nabla}^2 \bar{A} + \epsilon_0 \mu_0 \frac{\partial^2 \bar{A}}{\partial t^2} = 0$$

$$(27) \quad -\bar{\nabla}^2 \varphi + \epsilon_0 \mu_0 \frac{\partial^2 \varphi}{\partial t^2} = 0$$

We now choose $\varphi = 0$ (Coulomb gauge), and from (22) we see:

$$(28) \quad \bar{\nabla} \cdot \bar{A} = 0$$

So, the wave equation for our vector potential is:

$$(29) \quad -\bar{\nabla}^2 \bar{A} + \epsilon_0 \mu_0 \frac{\partial^2 \bar{A}}{\partial t^2} = 0$$

The solutions to this equation are plane waves.

$$(30) \quad \bar{A} = \bar{A}_0 \sin(\omega t - \bar{k} \cdot \bar{r} + \alpha) \quad \alpha : \text{phase}$$

$$(31) \quad = \bar{A}_0 \cos(\omega t - \bar{k} \cdot \bar{r} + \alpha')$$

\bar{k} is the wave vector which points along the direction of propagation and has a magnitude:

$$(32) \quad k^2 = \omega^2 \mu_0 \epsilon_0 = \omega^2 / c^2$$

Since (28) $\bar{\nabla} \cdot \bar{A} = 0$

$$-\bar{k} \cdot \bar{A}_0 \cos(\omega t - \bar{k} \cdot \bar{r} + \alpha) = 0$$

$$(33) \quad \therefore \bar{k} \cdot \bar{A}_0 = 0 \quad \bar{k} \perp \bar{A}_0$$

A_0 is the direction of the potential \rightarrow polarization. From (15) and (16), we see that for $\varphi = 0$: