Properties of Operators

1) The inverse of $\hat{A}$ (written $\hat{A}^{-1}$) is defined by

$$\hat{A}^{-1}\hat{A} = \hat{A}\hat{A}^{-1} = \mathbf{I} \tag{1}$$

2) The transpose of $\hat{A}$ (written A^T) is a matrix with elements inverted about the diagonal

$$\left(A^T\right)_{nq} = A_{qn} \tag{2}$$

If $A^T = -A$ then the matrix is antisymmetric.

3) The trace of $\hat{A}$ is defined as the sum over the diagonal elements of $\hat{A}$

$$Tr\left(\hat{A}\right) = \sum_q A_{qq} \tag{3}$$

4) The Hermitian Adjoint of $\hat{A}$, written $\hat{A}^\dagger$, is

$$\begin{aligned} \hat{A}^\dagger &= \left(\hat{A}^T\right)^* \\ \left(\hat{A}^\dagger\right)_{nq} &= \left(\hat{A}_{qn}\right)^* \end{aligned} \tag{4}$$

5) $\hat{A}$ is Hermitian if

$$\begin{aligned} \hat{A}^\dagger &= \hat{A} \\ \left(\hat{A}^T\right)^* &= A \end{aligned} \tag{5}$$

If $\hat{A}$ is Hermitian, then $\hat{A}^n$ is Hermitian and $e^{\hat{A}}$ is Hermitian. For a Hermitian operator, $\left\langle \psi \mid \hat{A}\varphi \right\rangle = \left\langle \psi \hat{A} \middle| \varphi \right\rangle$. Expectation values of Hermitian operators are real, so all physical observables are associated with Hermitian operators.

6) $\hat{A}$ is a unitary operator if the Hermitian adjoint is also the inverse operator

$$\begin{aligned} &\hat{A}^\dagger = \hat{A}^{-1} \\ &\left(\hat{A}^T\right)^* = \hat{A}^{-1} \\ &\hat{A}\hat{A}^\dagger = 1 \qquad \Rightarrow \qquad \left(\hat{A}\hat{A}^\dagger\right)_{nq} = \delta_{nq} \end{aligned} \tag{6}$$

If $\hat{A}$ is Hermitian, then $e^{i\hat{A}}$ is unitary.