MATH & PHYSICS REFERENCE MATERIAL

Differential Equations

Equation	Solution
Damped Harmonic Oscillator $a\ddot{x}+b\dot{x}+cx=0$	$x=e^{-(b/2a)t}\left(A\cos\mu t+B\sin\mu t\right)$ where A and B are constants and $\mu=\frac{1}{2a}\sqrt{4ac-b^2}$
Driven Harmonic Oscillator $m\ddot{x}+b\dot{x}+kx=F_0\cos(\omega t)$ $\ddot{x}+2\gamma\dot{x}+\omega_0^2x=\frac{F_0}{m}\cos(\omega t)$ $\gamma=b/2m \quad \omega_0=\sqrt{k/m}$	$x(t)=\frac{F_0/m}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}}\sin(\omega t+\beta)$ where $\tan\beta=\omega_0^2-\omega^2/2\gamma\omega$
$\dot{y}+ay=be^{i\alpha t}$	$y(t)=Ae^{-at}+\frac{be^{i\alpha t}}{a+i\alpha}$
$\frac{\partial y}{\partial t}=r\,y(t)+h(t)$	$y(t)=e^{rt}\int_0^t e^{-rt'}h(t')\,dt'+Ce^{rt}$ where r is a constant and C is a constant of integration.

Integrals

Gaussian Integrals	
$\int_{-\infty}^{+\infty} dt\, e^{-\alpha t^2} = \sqrt{\frac{\pi}{\alpha}}$ $\int_0^{\infty} dt\, t\, e^{-\alpha t^2} = \frac{1}{2\alpha}$ $\int_{-\infty}^{+\infty} dt\, t^2\, e^{-\alpha t^2} = \sqrt{\frac{\pi}{4\alpha^3}}$ $\int_0^{\infty} dt\, t^{2n+1}\, e^{-\alpha t^2} = \frac{n!}{2a^{n+1}}$ $\quad n\,is\ positive\ integer$	$\int_{-\infty}^{+\infty} dt\, e^{-(at^2+bt+c)} = \sqrt{\frac{\pi}{a}} \exp\left[\frac{b^2}{4a} - c\right]$ $\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx\, e^{-(ax^2+2bxy+cy^2)} = \frac{\pi}{\sqrt{ac-b^2}}$ $\int_{-\infty}^{+\infty} dt\, t^2\, e^{-\alpha t^2} = -\frac{d}{d\alpha}\int_{-\infty}^{+\infty} dt\, e^{-\alpha t^2}$
Completing the square:	
$\int_{-\infty}^{+\infty} dt \exp\left(-\alpha\left(t-\frac{ix}{2\alpha}\right)^2\right)$ $= \int_{-\infty}^{+\infty} dt\, e^{-\alpha t^2 + ixt + x^2/4\alpha} = \sqrt{\frac{\pi}{\alpha}}$	
Error Function	
$erf\,(x) = \sqrt{\frac{4}{\pi}}\int_0^x dt\, e^{-t^2}$ $\mathrm{erf}(-x) = -\mathrm{erf}(x)$ $\mathrm{erfc}(x) = 1-\mathrm{erf}(x)$ $\frac{d}{dx} erf\,(x) = \sqrt{\frac{4}{\pi}}\, e^{-x^2}$ $\int_0^x dt\, e^{-\alpha t^2} = \sqrt{\frac{\pi}{4\alpha}}\, \mathrm{erf}\left(\frac{x}{\sqrt{\alpha}}\right)$ $\int_0^x dt\, t^2\, e^{-\alpha t^2} = \frac{2}{\sqrt{\pi}}\, x\, e^{-\alpha x^2} + \sqrt{\frac{\pi}{4\alpha^3}}\, \mathrm{erf}\left(\frac{x}{\sqrt{\alpha}}\right)$	$\int_0^{\infty} dt\, e^{-(at^2+2bt+c)} = \frac{1}{2}\sqrt{\frac{\pi}{a}}\, e^{\frac{b^2-ac}{a}}\, \mathrm{erfc}\left(\frac{b}{\sqrt{a}}\right)$ $\int_{-\infty}^{\infty} dt\, e^{-(at^2+2bt)} = \sqrt{\frac{\pi}{a}}\, e^{\frac{-b^2}{a}}$ $\int_0^{\infty} dt\, e^{-(at^2+2bt)} = \frac{1}{2}\sqrt{\frac{\pi}{a}}\, e^{b^2/a}\, \mathrm{erfc}\left(\frac{b}{\sqrt{a}}\right)$ $\int_0^{\infty} dt\, e^{-at^2}\, \mathrm{erf}(bx) = \sqrt{\frac{1}{a\pi}}\, \mathrm{atan}\,\frac{1}{a}$

Dirac Delta Function

Definitions and Identities

$$\delta(x)=\begin{cases}0 & x\neq 0\\ 1 & x=0\end{cases}$$

$$\int_{-\infty}^{+\infty}\delta(x-a)\,f(x)\,dx=f(a)$$

$$\delta(x)=\delta(-x)$$

$$\int_{-\infty}^{+\infty}\delta(x)dx=1$$

$$\delta(x)=\frac{d}{dx}\Theta(x)$$ Delta function is the derivative of the unit step function

$$x\frac{d}{dx}\delta(x)=-\delta(x)$$

$$\delta(ax)=\frac{1}{|\mathrm{a}|}\delta(x)\qquad\rightarrow\ \delta(E)=\frac{1}{\hbar}\delta(\omega)$$

Representations of the delta function

$$\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{ikx}dk$$

$$\delta(x)=\lim_{a\to\infty}\frac{2\sin^2\tfrac{1}{2}ax}{\pi ax^2}$$

$$\delta(x)=\lim_{a\to\infty}\frac{\sin ax}{\pi x}$$

$$\delta(x)=\lim_{a\to 0}\mathrm{Im}\left[\frac{1}{\pi\left[x-ia\right]}\right]$$

$$\delta(x)=\lim_{a\to 0}\frac{a}{\pi\left[x^2+a^2\right]}$$

$$\delta(x)=\lim_{a\to\infty}\sqrt{\frac{a}{\pi}}\,e^{-ax^2}$$

Spherical Coordinates

$$\delta(\Omega-\Omega_0)=\frac{\delta(\theta-\theta_0)\delta(\phi-\phi_0)}{\sin\theta}=\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}Y_{\ell,m}^{*}\left(\theta,\phi\right)Y_{\ell,m}\left(\theta_0,\phi_0\right)$$

$$\int_0^{\pi}d\theta\int_0^{2\pi}d\phi\ \delta(\theta-\theta_0)\delta(\phi-\phi_0)=1$$

$$\int d\Omega\equiv\frac{1}{4\pi}\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi\qquad\qquad\int\delta(\Omega-\Omega_0)d\Omega=1$$

Commutators and Exponential Operators

$$\left[\hat{A},\hat{B}\right]=\hat{A}\hat{B}-\hat{B}\hat{A}$$

$$\left[\hat{A},\hat{B}\right]=-\left[\hat{B},\hat{A}\right]$$

$$\left[\hat{A},\hat{B}+\hat{C}\right]=\left[\hat{A},\hat{B}\right]+\left[\hat{B},\hat{C}\right]$$

$$\left[\hat{A},\hat{B}\hat{C}\right]=\left[\hat{A},\hat{B}\right]\hat{C}+\hat{B}\left[\hat{A},\hat{C}\right]$$

$$\left[\hat{A},\hat{B}\right]^{\dagger}=\left[\hat{B}^{\dagger},\hat{A}^{\dagger}\right]$$

$$\left[\left[\hat{C},\hat{B}\right],\hat{A}\right]=\hat{C}\hat{B}\hat{A}-\hat{B}\hat{C}\hat{A}-\hat{A}\hat{C}\hat{B}+\hat{A}\hat{B}\hat{C}$$

$$\left[\left[\hat{C},\hat{B}\right],\hat{A}\right]=\left[\left[\hat{A},\hat{B}\right],\hat{C}\right]$$

$$\left[\hat{A},\left[\hat{B},\hat{C}\right]\right]+\left[\hat{B},\left[\hat{C},\hat{A}\right]\right]+\left[\hat{C},\left[\hat{A},\hat{B}\right]\right]=0$$

$$e^{\hat{A}+\hat{B}}=e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}\left[\hat{A},\hat{B}\right]}$$ (…if $\hat{A}$ and $\hat{B}$ commute with $\left[\hat{A},\hat{B}\right]$)

$$e^{\hat{A}}\,e^{\hat{B}}=e^{\hat{B}}\,e^{\hat{A}}\,e^{-\left[\hat{B},\hat{A}\right]}$$

$$e^{\lambda a^{\dagger}+\mu a}=e^{\lambda a^{\dagger}}\,e^{\mu a}\,e^{\frac{1}{2}\lambda\mu}$$ (…where λ and μ are constants)

$$e^{i\hat{G}\lambda}\hat{A}\,e^{-i\hat{G}\lambda}=\hat{A}+i\lambda\left[\hat{G},\hat{A}\right]+\left(\frac{i^{2}\lambda^{2}}{2!}\right)\left[\hat{G},\left[\hat{G},\hat{A}\right]\right]+\ldots+\left(\frac{i^{n}\lambda^{n}}{n!}\right)\left[\hat{G},\left[\hat{G},\left[\hat{G}\ldots\left[\hat{G},\hat{A}\right]\right]\right]\ldots\right]+\ldots$$

$$e^{\hat{A}}e^{\hat{B}}=\exp\left[\hat{A}+\hat{B}+\frac{1}{2}\left[\hat{A},\hat{B}\right]+\frac{1}{12}\left(\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]+\left[\hat{A},\left[\hat{B},\hat{B}\right]\right]\right)+\cdots\right]$$

Raising and Lowering Operators

$$\left[a, a^\dagger\right] = 1$$

$$\left[a, \left(a^\dagger\right)^n\right] = \left[a, a^\dagger\right] n \left(a^\dagger\right)^{n-1} = n \left(a^\dagger\right)^{n-1}$$

$$\left[a^\dagger, a^n\right] = \left[a^\dagger, a\right] n a^{n-1} = -n a^{n-1}$$

$$x = \sqrt{\frac{\hbar}{2m\omega}} \left(a + a^\dagger\right)$$

$$p = i\sqrt{\frac{m\hbar\omega}{2}} \left(a^\dagger - a\right)$$

$$p(t) = i\sqrt{\frac{m\hbar\omega_0}{2}} \left(a^\dagger e^{i\omega_0 t} - a\, e^{-i\omega_0 t}\right)$$

$$x(t) = \sqrt{\frac{\hbar}{2m\omega}} \left(a^\dagger e^{i\omega_0 t} + a\, e^{-i\omega_0 t}\right)$$

Harmonic Oscillator Identities

$$\hat{H} = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \tfrac{1}{2}\right)$$

$$\left|0\right\rangle = N \exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

$$\left|n\right\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger\right)^n \left|0\right\rangle$$

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{a}\right] = -i\omega_0 \hat{a} \qquad \rightarrow \qquad \hat{a}(t) = \hat{a} e^{-i\omega_0 t}$$

$$\frac{d\hat{a}^\dagger}{dt} = \frac{i}{\hbar}\left[\hat{H}, \mathrm{a}^\dagger\right] = i\omega_0 \mathrm{a}^\dagger \qquad \rightarrow \qquad \hat{a}^\dagger(t) = \mathrm{a}^\dagger e^{i\omega_0 t}$$

Thermally averages:

$$\langle a_\alpha a_\alpha^\dagger \rangle = \bar{n}_\alpha + 1 \qquad \bar{n}_\alpha = \left(e^{\beta\hbar\omega_\alpha} - 1\right)^{-1}$$

$$\langle a_\alpha^\dagger a_\alpha \rangle = \bar{n}_\alpha$$

$$\bar{n} = \langle n \rangle_{eq} = \langle a^\dagger a \rangle = \frac{1}{e^{\beta\hbar\omega} - 1} = \frac{e^{-\beta\hbar\omega/2}}{e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}}$$

$$\bar{n} + 1 = \langle a a^\dagger \rangle = \frac{1}{1 - e^{-\beta\hbar\omega}} = \frac{e^{\beta\hbar\omega/2}}{e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}}$$

$$2\bar{n} + 1 = \langle a^\dagger a + a a^\dagger \rangle = \frac{e^{\beta\hbar\omega} + 1}{e^{\beta\hbar\omega} - 1} = \frac{e^{\beta\hbar\omega/2} + e^{-\beta\hbar\omega/2}}{e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}} = \coth\left(\frac{\beta\hbar\omega}{2}\right)$$

if $H = \hbar\omega_\alpha \left(a_\alpha^\dagger a_\alpha + \frac{1}{2}\right)$ then:

$$e^{iHt}\, a_\alpha\, e^{-iHt} = a_\alpha e^{-i\omega_\alpha t}$$

$$e^{iHt}\, a_\alpha^\dagger\, e^{-iHt} = a_\alpha^\dagger e^{+i\omega_\alpha t}$$

$$\begin{aligned} C_{xx}(t) = \langle x(t)\, x(0) \rangle &= \left[(\bar{n}+1)e^{-i\omega_0 t} + \bar{n}\, e^{i\omega_0 t}\right] \\ &= \left[(2\bar{n}+1)\cos\omega_0 t + i\sin\omega_0 t\right] \\ &= \left[\coth(\beta\hbar\omega_0/2)\cos\omega_0 t + i\sin\omega_0 t\right] \end{aligned}$$

Exponential Identities

For operators linear in the harmonic oscillator coordinate, $\hat{A} \propto \left(a + a^\dagger\right)$: $\quad \langle e^{\hat{A}} \rangle = e^{\frac{1}{2}\langle \hat{A}^2 \rangle}$

$$\left\langle e^{\alpha_1 \hat{a} + \beta_1 \hat{a}^\dagger} e^{\alpha_2 \hat{a} + \beta_2 \hat{a}^\dagger} \right\rangle = e^{(\alpha_1+\alpha_2)(\beta_1+\beta_2)\left(n+\frac{1}{2}\right) + \frac{1}{2}(\alpha_1\beta_2 - \beta_1\alpha_2)}$$

$$e^{\lambda a^\dagger + \mu a} = e^{\lambda a^\dagger}\, e^{\mu a}\, e^{\frac{1}{2}\lambda\mu}$$