EINSTEIN B COEFFICIENT AND ABSORPTION CROSS-SECTION

The rate of absorption induced by the field is

$$w_{k\ell}(\omega) = \frac{\pi}{2\hbar^2} \left| E_0(\omega) \right|^2 \left| \langle k | \hat{\varepsilon} \cdot \bar{\mu} | \ell \rangle \right|^2 \delta(\omega_{k\ell} - \omega) \tag{1}$$

The rate is clearly dependent on the strength of the field. The variable that you can most easily measure is the intensity *I*, the energy flux through a unit area, which is the time-averaged value of the Poynting vector, *S*

$$S = \frac{c}{4\pi}\left(\bar{E} \times \bar{B}\right) \tag{2}$$

$$I = \langle S \rangle = \frac{c}{4\pi}\left\langle \bar{E}^2 \right\rangle = \frac{c}{8\pi} E_0^2 . \tag{3}$$

Using this we can write

$$w_{k\ell} = \frac{4\pi^2}{3c\hbar^2} I(\omega) \left| \langle k | \hat{\varepsilon} \cdot \bar{\mu} | \ell \rangle \right|^2 \delta(\omega_{k\ell} - \omega) , \tag{4}$$

where I have also made use of the uniform distribution of polarizations applicable to an <u>isotropic field</u>: $\left|\bar{E}_0 \cdot \hat{x}\right| = \left|\bar{E}_0 \cdot \hat{y}\right| = \left|\bar{E}_0 \cdot \hat{z}\right| = \frac{1}{3}\left|E_0\right|^2$. An equivalent representation of the amplitude of a monochromatic field is the energy density

$$U = \frac{I}{c} = \frac{1}{8\pi} E_0^2 . \tag{5}$$

which allows the rates of transition to be written as

$$w_{k\ell} = B_{k\ell}\, U(\omega_{k\ell}) \tag{6}$$

The first factor contains the terms in the matter that dictate the absorption rate. B is independent of the properties of the field and is called the Einstein B coefficient

$$B_{k\ell} = \frac{4\pi^2}{3\hbar^2} \left|\mu_{k\ell}\right|^2 . \tag{7}$$

You may see this written elsewhere as $B_{k\ell} = \left(2\pi/3\hbar^2\right)\left|\mu_{k\ell}\right|^2$, which holds when the energy density of a wave is expressed in Hz instead of angular frequency.

If we associate the energy density with a number of photons *N*, then U can also be written in a quantum form

$$N\hbar\omega = \frac{E_0^2}{8\pi} \qquad U = N\frac{\hbar\omega^3}{\pi^2 c^3}\,. \tag{8}$$

Now let's relate the rates of absorption to a quantity that is directly measured, an absorption cross-section α:

$$\begin{aligned}\alpha &= \frac{total\ energy\ absorbed\ /\ unit\ time}{total\ incident\ \text{int}ensity\left(energy\ /\ unit\ time\ /\ area\right)} \\ &= \frac{\hbar\omega \cdot w_{k\ell}}{I} = \frac{\hbar\omega \cdot B_{k\ell}U\left(\omega_{k\ell}\right)}{cU\left(\omega_{k\ell}\right)} \\ &= \frac{4\pi^2}{\hbar c}\left|\mu_{k\ell}\right|^2 = \frac{\hbar\omega}{c}B_{k\ell}\end{aligned} \tag{9}$$

More generally, you may have a frequency-dependent absorption coefficient $\alpha\left(\omega\right) \propto B_{k\ell}\left(\omega\right)$ $= B_{k\ell}\ g\left(\omega\right)$ where $g(\omega)$ is a unit normalized lineshape function. The golden rule rate for absorption also gives the same rate for stimulated emission. Given two levels $\left|m\right\rangle\ and\ \left|n\right\rangle$:

$$w_{nm} = w_{mn}$$

$$B_{nm}U\left(\omega_{nm}\right) = B_{nm}U\left(\omega_{nm}\right) \qquad \text{since } U\left(\omega_{nm}\right) = U\left(\omega_{mn}\right) \tag{10}$$

$$B_{nm} = B_{mn}$$

The absorption probability per unit time equals the stimulated emission probability per unit time. Also, the cross-section for absorption is equal to an equivalent cross-section for stimulated emission, $\left(\alpha_A\right)_{nm} = \left(\alpha_{SE}\right)_{mn}$.